Rewriting Systems in a Nutshell

Non-Terminating Systems

- Non-terminating systems can be meaningful
  - Modelling reactive systems, e.g. by process calculi
  - Approximation algorithms which enhance the accuracy of the
    approximation with each iteration, e.g. computing $\pi$
  - Specification of infinite data structures, e.g. streams

Example: list of all natural numbers

$$\text{list of all natural numbers: } R_{\text{list}} = \{x + 0 \rightarrow x, 0 + y \rightarrow 0, z + (x + y) \rightarrow z + x + y\}$$

Most important properties of rewriting systems

- Confluence: ensures that normal forms, i.e. results of computations, are unique
- Termination: ensures that every computation eventually halts, i.e. reaches a normal form

For example, $R_{\text{list}}$ is confluent and terminating.

Contributions of the Thesis

Partial Order Model of Infinitary Rewriting

- The partial order is known to form a complete semilattice
  - Limit inferior is always defined
  - Every reduction sequence converges
- Also allows to distinguish between weak and strong convergence
  - Weak convergence: limit inferior of the terms of the reduction
  - Strong convergence: limit inferior of the contexts of the reduction

Transfinite Abstract Reduction Systems

- Both the metric and the partial order approach of transfinite reductions were analysed on an abstract level:
  - Transfinite reductions in both models have similar properties as finite reductions
  - Transfinite reductions in both models have similar properties as
    finite reductions

$\text{weak convergence: limit inferior of the terms of the reduction}$

$\text{strong convergence: limit inferior of the contexts of the reduction}$

Infinitary Term Rewriting

- Infinitary term rewriting allows reductions of transfinite length:
  - Terms are endowed with a complete metric in order to formalise the
    convergence of infinite reductions.
  - Metric distance between terms is inversely proportional to the
    shallowest depth at which they differ

- Two different variants of convergence are considered:
  - Weak convergence: convergence in the metric space
  - Strong convergence: convergence in the metric space + depth of where
    the rewrite rules are applied tends to infinity

Example for weak convergence

System with single rule $f(g(a)) \rightarrow f(g(g(a)))$ induces infinite reduction weakly converging to $f(g(g(g(\ldots)))) = f(g^\omega)$.

Example for strong convergence

System with single rule $g(a) \rightarrow g(g(a))$ induces infinite reduction strongly converging to $f(g(g(g(\ldots)))) = f(g^\omega)$.

Results and Perspective

- The introduced partial order infinitary term rewriting has more
  advantageous properties and is more intuitive than the
  well-established metric model

- The devised complete semilattice and complete metric on term
  graphs allow infinitary term rewriting and can serve as a tool
  for investigating the semantics of term rewriting systems.

Future Work

- Further investigation of infinitary term graph rewriting
  - Might help finding (common) properties of rational term rewriting
  - Generalisation of congruence results of finitary case
- Identifying which class of infinitary term rewriting infinitary term graph rewriting can simulate
- Finding more heuristics to transform term rewriting systems to term
  graph rewriting systems in order to implement infinitary term
  rewriting
- Using other term graph rewriting approaches (equational and
  double-pushout approach seem promising) to simulate infinitary
  term graph in the partial order model
- Employing the partial order on term graphs to generalise Böhm
  trees (of term rewriting systems) to Böhm graphs (of term rewriting
  systems)

Contact: Patrick Bahr, pa-bahr@ac.tu-wien.ac

Infinitary Rewriting

Theory and Applications