Infinitary Term Graph Rewriting is Simple, Sound and Complete

Patrick Bahr
paba@diku.dk

University of Copenhagen
Department of Computer Science

23rd International Conference on Rewriting Techniques and Applications,
Nagoya, Japan, March May 30 – June 1, 2012
Infinitary Rewriting vs. Term Graph Rewriting

Pick one to avoid the other.
Infinitary Rewriting vs. Term Graph Rewriting

Pick one to avoid the other.

Pick term graph rewriting

- finite representation of infinite terms (via cycles)
- finite representation of infinite rewrite sequences

\[ f \rightarrow g \rightarrow h \rightarrow b \]
**Infinitary Rewriting vs. Term Graph Rewriting**

Pick one to avoid the other.

**Pick term graph rewriting**
- finite representation of infinite terms (via cycles)
- finite representation of infinite rewrite sequences

**Pick infinitary rewriting**
- avoid dealing with term graphs
- work on the *unravelling* instead

---

**Diagram**

```
    f
   / \   \\
  g   h
 /     \        \\
 b      c
```

---
A common formalism

study correspondences between infinitary TRSs and finitary GRSs
Infinitary Term Graph Rewriting – What is it for?

A common formalism study **correspondences** between infinitary TRSs and finitary GRSs

**Lazy evaluation**
- infinitary term rewriting **only covers non-strictness**
- however: lazy evaluation = non-strictness + **sharing**
Infinitary Term Graph Rewriting – What is it for?

A common formalism study correspondences between infinitary TRSs and finitary GRSs

Lazy evaluation

- infinitary term rewriting only covers non-strictness
- however: lazy evaluation = non-strictness + sharing

towards infinitary lambda calculi with letrec

- Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
- the calculus is non-confluent
- but there is a notion of infinite normal forms
Approach

Previous approach (RTA ’11)

- weak convergence
- two modes of convergence: metric & partial order
Approach

Previous approach (RTA ’11)

- weak convergence
- two modes of convergence: metric & partial order

result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting (sorta kinda)
Approach

Previous approach (RTA ’11)

- weak convergence
- two modes of convergence: metric & partial order
- result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting (sorta kinda)
- problem: complicated; difficult to analyse; completeness ??
Approach

Previous approach (RTA ’11)

- weak convergence
- two modes of convergence: metric & partial order
- result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting (sorta kinda)
- problem: complicated; difficult to analyse; completeness ??

Our new approach

- strong convergence
- two modes of convergence: metric & partial order
Approach

Previous approach (RTA ’11)

- weak convergence
- two modes of convergence: metric & partial order
- result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting (sorta kinda)
- problem: complicated; difficult to analyse; completeness ??

Our new approach

- strong convergence
- two modes of convergence: metric & partial order
- but: simpler (ignoring the sharing as much as possible)
Approach

**Previous approach (RTA ’11)**

- weak convergence
- two modes of convergence: metric & partial order
- result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting (sorta kinda)
- problem: complicated; difficult to analyse; completeness ??

**Our new approach**

- strong convergence
- two modes of convergence: metric & partial order
- but: simpler (ignoring the sharing as much as possible)
- result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting
  - completeness w.r.t. infinitary term rewriting
Approach

Previous approach (RTA ’11)

- weak convergence
- two modes of convergence: metric & partial order
- result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting (sorta kinda)
- problem: complicated; difficult to analyse; completeness ??

Our new approach

- strong convergence \(\Rightarrow\) independence from the rewriting formalism
- two modes of convergence: metric & partial order
- but: simpler (ignoring the sharing as much as possible)
- result:
  - correspondence between metric & partial order approach
  - soundness w.r.t. infinitary term rewriting
  - completeness w.r.t. infinitary term rewriting
Outline

1. Introduction
   - Goals
   - A Different Approach

2. Modes of Convergence on Term Graphs
   - Metric Approach
   - Partial Order Approach
   - Metric vs. Partial Order Approach
Metric Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms:

\[ d(s, t) = 2^{-\text{sim}(s, t)} \]

\[ \text{sim}(s, t) = \text{maximum depth } d \text{ s.t. } s \text{ and } t \text{ coincide up to depth } d \]
Metric Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a **complete metric** in order to formalise the **convergence** of infinite reductions.

- metric distance between terms:

  \[ d(s, t) = 2^{-\text{sim}(s, t)} \]

  \[ \text{sim}(s, t) = \text{maximum depth } d \text{ s.t. } s \text{ and } t \text{ coincide up to depth } d \]

Strong convergence via metric \( d \) and redex depth

- convergence in the metric space \((T\infty(\Sigma), d)\)

  \[ \rightsquigarrow \text{depth of the differences between the terms has to tend to infinity} \]
Metric Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms:

\[ d(s, t) = 2^{-\text{sim}(s, t)} \]

\[ \text{sim}(s, t) = \text{maximum depth } d \text{ s.t. } s \text{ and } t \text{ coincide up to depth } d \]

Strong convergence via metric \( d \) and redex depth

- convergence in the metric space \((\mathcal{T}^\infty(\Sigma), d)\)
- depth of the differences between the terms has to tend to infinity
- depth of redexes has to tend to infinity
Example: Metric Convergence in TRSs

\[
\text{from} \quad \rightarrow \quad 0
\]

\[
\text{from}(x) \rightarrow x :: \text{from}(s(x))
\]
Example: Metric Convergence in TRSs

```
from
  ↓
  0
```

```
::
  ↓
from
  ↓
  0
  ↓
  1
```

\[
\text{from}(x) \rightarrow x :: \text{from}(s(x))
\]
Example: Metric Convergence in TRSs

\[ \text{from} \quad \downarrow \quad + \]

\[
\begin{array}{c}
\triangledown \\
0 \\
\end{array}
\quad \\
\begin{array}{c}
\triangledown \\
0 \\
\end{array}
\quad \\
\begin{array}{c}
\triangledown \\
1 \\
\end{array}
\quad \\
\begin{array}{c}
\triangledown \\
2 \\
\end{array}
\]

\[ \text{from} \quad \downarrow \quad \cdots \quad \downarrow \quad \text{from} \]

\[ \text{from}(x) \rightarrow x :: \text{from}(s(x)) \]
Example: Metric Convergence in TRSs

\[ \text{from}(x) \rightarrow x :: \text{from}(s(x)) \]
Example: Metric Convergence in TRSs

\[ \text{from}(x) \rightarrow x :: \text{from}(s(x)) \]
Example: Metric Convergence in TRSs

\[ \text{from}(x) \rightarrow x :: \text{from}(s(x)) \]
Example: Metric Convergence in TRSs

\[ \text{from} \]

\[
\begin{align*}
\text{from} & \quad \rightarrow \quad + \\
0 & \quad \downarrow \\
\end{align*}
\]

\[
\begin{align*}
\text{from} & \quad \rightarrow \quad + \\
0 & \quad \downarrow \\
\vdots & \quad \downarrow \quad \downarrow \\
1 & \quad \downarrow \\
\vdots & \quad \downarrow \\
2 & \quad \downarrow \\
\vdots & \quad \downarrow \\
3 & \quad \downarrow \\
\vdots & \quad \downarrow \\
4 & \\
\end{align*}
\]

\[
\text{from}(x) \rightarrow x :: \text{from}(s(x))
\]
Example: Metric Convergence in TRSs

\[ \text{from}(x) \rightarrow x :: \text{from}(s(x)) \]
Example: Metric Convergence in TRSs

\[ \text{from}(x) \rightarrow x :: \text{from}(s(x)) \]
Example: Metric Convergence in TRSs

\[ \text{from}(x) \rightarrow x :: \text{from}(s(x)) \]
A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.
The truncation $g^d$ is obtained from $g$ by

- relabelling all nodes at depth $d$ with $\bot$, and
- removing all nodes that thus become unreachable from the root.
A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.

Truncation of term graphs

The truncation $g^{\dagger d}$ is obtained from $g$ by

- relabelling all nodes at depth $d$ with ⊥, and
- removing all nodes that thus become unreachable from the root.

The simple metric on term graphs

$$d^{\dagger}(g, h) = 2^{-\text{sim}^{\dagger}(g, h)}$$

Where $\text{sim}^{\dagger}(g, h) = \text{maximum depth } d \text{ s.t. } g^{\dagger d} \cong h^{\dagger d}$. 
A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.

Truncation of term graphs

The truncation $g^\dagger d$ is obtained from $g$ by
- relabelling all nodes at depth $d$ with ⊥, and
- removing all nodes that thus become unreachable from the root.

The simple metric on term graphs

$$d^\dagger(g, h) = 2^{-\text{sim}^\dagger(g, h)}$$

Where $\text{sim}^\dagger(g, h) = \text{maximum depth } d \text{ s.t. } g^\dagger d \simeq h^\dagger d$.

Strong convergence via metric $d^\dagger$ and redex depth

- convergence in the metric space $(G^\infty_C(\Sigma), d^\dagger)$
- depth of redexes has to tend to infinity
Soundness & Completeness

Theorem (soundness of metric convergence)
For every left-linear, left-finite GRS \( \mathcal{R} \) we have

\[
\forall x, y \in \mathcal{R}, \quad d(x, y) = 0 \implies x = y
\]

Completeness property

\[
\forall x, y \in \mathcal{R}, \quad d(x, y) = 0 \implies x = y
\]

[Kennaway et al., 1994]
Soundness & Completeness

soundness of metric convergence

\[ \mathcal{R} \xrightarrow{g} m \xrightarrow{m} h \]

\[ \mathcal{U}(\cdot) \]

\[ \mathcal{U}(\mathcal{R}) \]
Soundness & Completeness

soundness of metric convergence

\[ \mathcal{R} \xrightarrow{g} m \xrightarrow{h} \]
\[ \mathcal{U}(\cdot) \]
\[ \mathcal{U}(\mathcal{R}) \xrightarrow{s} m \xrightarrow{t} \]

\[ \text{Kennaway et al., 1994} \]
Soundness & Completeness

**Theorem (soundness of metric convergence)**

For every left-linear, left-finite GRS \( R \) we have

\[
\begin{array}{ccc}
\mathcal{U}(\cdot) & \mathcal{U}(R) & s \\
\downarrow & \downarrow & m \\
\mathcal{U}(\cdot) & \mathcal{U}(R) & t \\
\end{array}
\]

\( R \xrightarrow{g} h \)

[Kennaway et al., 1994]
Soundness & Completeness

Theorem (soundness of metric convergence)
For every left-linear, left-finite GRS $\mathcal{R}$ we have

Let $g : \mathcal{R} \rightarrow m$,

Let $h : m \rightarrow h$,

Let $t : m \rightarrow t$,

Completeness property

Let $s : \mathcal{U}(\mathcal{R}) \rightarrow m$,

Let $t : m \rightarrow t$,

Kennaway et al., 1994
Soundness & Completeness

Theorem (soundness of metric convergence)
For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\begin{array}{cccccc}
\mathcal{R} & g & \overset{m}{\longrightarrow} & h \\
\mathcal{U}(\cdot) & \downarrow & & \downarrow & \mathcal{U}(\cdot) \\
\mathcal{U}(\mathcal{R}) & s & \overset{m}{\longrightarrow} & t \\
\end{array}
\]

Completeness property

\[
\begin{array}{cccccc}
\mathcal{U}(\mathcal{R}) & s & \overset{m}{\longrightarrow} & t \\
\mathcal{U}(\cdot) & \uparrow & & \uparrow & \mathcal{U}(\cdot) \\
\mathcal{R} & g & \overset{m}{\longrightarrow} & h \\
\end{array}
\]
Soundness & Completeness

**Theorem (soundness of metric convergence)**

For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\begin{array}{ccccccc}
\mathcal{R} & g & m & h \\
\mathcal{U}(\cdot) & \downarrow & & \downarrow \\
\mathcal{U}(\mathcal{R}) & s & m & t \\
\end{array}
\]

**Completeness property**

\[
\begin{array}{ccccccc}
\mathcal{U}(\mathcal{R}) & s & m & t \\
\mathcal{U}(\cdot) & \downarrow & \uparrow \\
\mathcal{U}(\cdot) & \downarrow & \uparrow \\
\mathcal{R} & g & m & \times \\
\end{array}
\]

[Kennaway et al., 1994]
Soundness & Completeness

Theorem (soundness of metric convergence)
For every left-linear, left-finite GRS $\mathcal{R}$ we have

$\mathcal{R} \xrightarrow{g} \mathcal{U}(\cdot) \xrightarrow{\mathcal{U}(\cdot)} \mathcal{U}(\mathcal{R}) \xrightarrow{s} m \xrightarrow{\mathcal{R}} \mathcal{U}(\cdot) \xrightarrow{\mathcal{U}(\cdot)} \mathcal{U}(\mathcal{R}) \xrightarrow{t} m \xrightarrow{\mathcal{R}} \mathcal{U}(\cdot) \xrightarrow{\mathcal{U}(\cdot)} \mathcal{U}(\mathcal{R}) \xrightarrow{t'} \mathcal{U}(\cdot) \xrightarrow{h} \mathcal{R}$

Completeness property

$\mathcal{U}(\mathcal{R}) \xrightarrow{s} \mathcal{U}(\cdot) \xrightarrow{\mathcal{U}(\cdot)} \mathcal{U}(\mathcal{R}) \xrightarrow{m} t \xrightarrow{\mathcal{R}} \mathcal{U}(\cdot) \xrightarrow{\mathcal{U}(\cdot)} \mathcal{U}(\mathcal{R}) \xrightarrow{m} t' \xrightarrow{\mathcal{R}} \mathcal{U}(\cdot) \xrightarrow{\mathcal{U}(\cdot)} \mathcal{U}(\mathcal{R}) \xrightarrow{h} \mathcal{R}$
Soundness & Completeness

Theorem (soundness of metric convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\begin{array}{cccccc}
\mathcal{R} & g & m & \rightarrow & \mathcal{R} & t \\
\mathcal{U}(\cdot) & \downarrow & & & \downarrow & \mathcal{U}(\cdot) \\
\mathcal{U}(\mathcal{R}) & s & m & \rightarrow & \mathcal{U}(\mathcal{R}) & t
\end{array}
\]

Completeness property

\[
\begin{array}{cccccc}
\mathcal{U}(\mathcal{R}) & s & m & \rightarrow & t & m & \rightarrow & t' \\
\mathcal{U}(\cdot) & \downarrow & & \downarrow & \mathcal{U}(\cdot) \\
\mathcal{U}(\cdot) & \downarrow & & \downarrow & \mathcal{U}(\cdot) \\
\mathcal{R} & g & m & \rightarrow & \mathcal{R} & t
\end{array}
\]

[Kennaway et al., 1994]
Outline

1 Introduction
   • Goals
   • A Different Approach

2 Modes of Convergence on Term Graphs
   • Metric Approach
   • Partial Order Approach
   • Metric vs. Partial Order Approach
Partial Order Infinitary Term Rewriting

<table>
<thead>
<tr>
<th>Partial order on terms</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>partial terms</strong>: terms with additional constant ( \perp ) (read as “undefined”)</td>
</tr>
<tr>
<td>partial order ( \leq_{\perp} ) reads as: “is less defined than”</td>
</tr>
<tr>
<td>( \leq_{\perp} ) is a <strong>complete semilattice</strong> (( = \text{cpo} + \text{glbs of non-empty sets} ))</td>
</tr>
</tbody>
</table>
Partial Order Infinitary Term Rewriting

Partial order on terms
- **partial terms**: terms with additional constant \( \bot \) (read as “undefined”)
- partial order \( \leq \bot \) reads as: “is less defined than”
- \( \leq \bot \) is a complete semilattice (\( = \text{cpo} + \text{glbs of non-empty sets} \))

Convergence
- formalised by the limit inferior:

\[
\lim_{l \to \alpha} \inf t_l = \bigcup_{\beta < \alpha} \bigcap_{\beta \leq l < \alpha} t_l
\]

- intuition: eventual persistence of nodes of the terms
- **weak convergence**: limit inferior of the terms of the reduction
Partial Order Infinitary Term Rewriting

**Partial order on terms**
- **partial terms**: terms with additional constant ⊥ (read as “undefined”)
- **partial order** \( \leq \) reads as: “is less defined than”
- \( \leq \) is a complete semilattice (= cpo + glbs of non-empty sets)

**Convergence**
- formalised by the limit inferior:
  \[
  \liminf_{\iota \to \alpha} t_{\iota} = \bigsqcup_{\beta < \alpha} \bigcap_{\beta \leq \iota < \alpha} t_{\iota}
  \]
- intuition: eventual persistence of nodes of the terms
- **weak convergence**: limit inferior of the terms of the reduction
- **strong convergence**: limit inferior of the contexts of the reduction
Partial Order Infinitary Term Rewriting

Partial order on terms

- **partial terms**: terms with additional constant \( \perp \) (read as “undefined”)
- partial order \( \leq \perp \) reads as: “is less defined than”
- \( \leq \perp \) is a complete semilattice (= cpo + glbs of non-empty sets)

Convergence

- formalised by the **limit inferior**:
  \[
  \liminf_{l \to \alpha} t_l = \bigsqcup_{l \leq \alpha} \bigcap_{\beta < \alpha} t_l
  \]
  - term obtained by replacing the redex with \( \perp \)
  - intuition: eventual persistence of nodes of the terms
  - **weak convergence**: limit inferior of the terms of the reduction
  - **strong convergence**: limit inferior of the contexts of the reduction
### Partial Order Convergence vs. Metric Convergence

#### Intuition of partial order convergence
- Subterms that break \( m \)-convergence do \( p \)-converge to \( \bot \).
- Every (continuous) reduction converges.

---

**Theorem (total \( p \)-convergence = \( m \)-convergence)**

For every reduction \( S \) in a TRS, the following equivalence holds:

\[
S : s \rightarrow p t \text{ total iff } S : s \rightarrow m t
\]

**Theorem (normalisation & confluence)**

Every orthogonal TRS is infinitarily normalising and infinitarily confluent w.r.t. strong \( p \)-convergence.
Partial Order Convergence vs. Metric Convergence

Intuition of partial order convergence
- subterms that break \( m \)-convergence do \( p \)-converge to \( \bot \)
- every (continuous) reduction converges

Theorem (total \( p \)-convergence = \( m \)-convergence)

For every reduction \( S \) in a TRS the following equivalence holds:

\[
S: s \xrightarrow{p} t \text{ total} \quad \iff \quad S: s \xrightarrow{m} t
\]
Partial Order Convergence vs. Metric Convergence

Intuition of partial order convergence
- subterms that break $m$-convergence do $p$-converge to $\bot$
- every (continuous) reduction converges

Theorem (total $p$-convergence = $m$-convergence)

For every reduction $S$ in a TRS the following equivalence holds:

$$S: s \xrightarrow{p} t \text{ total} \iff S: s \xrightarrow{m} t$$

Theorem (normalisation & confluence)

Every orthogonal TRS is infinitarily normalising and infinitarily confluent w.r.t. strong $p$-convergence.
A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order $\leq_{\bot}$ on term trees?
A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order \( \leq \) on term trees?

\( \perp \)-homomorphisms \( \phi: g \rightarrow \perp h \)

- homomorphism condition suspended on \( \perp \)-nodes
- allow mapping of \( \perp \)-nodes to arbitrary nodes
- same mechanism describing matching in term graph rewriting
A Partial Order on Term Graphs – How?

Specialise on terms
- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order $\leq_\bot$ on term trees?

$\bot$-homomorphisms $\phi: g \rightarrow_\bot h$
- homomorphism condition suspended on $\bot$-nodes
- allow mapping of $\bot$-nodes to arbitrary nodes
- same mechanism describing matching in term graph rewriting

Definition (Simple partial order $\leq_S_\bot$ on term graphs)
For all $g, h \in G^\infty(\Sigma_\bot)$, let $g \leq_S_\bot h$ iff there is some $\phi: g \rightarrow_\bot h$. 
Partial Order Convergence on Term Graphs

**Convergence**

- **Weak conv.:** limit inferior of the term graphs along the reduction.
- **Strong conv.:** limit inferior of the contexts along the reduction.

**Example**

```
context
```
Partial Order Convergence on Term Graphs

### Convergence

- **Weak conv.:** limit inferior of the term graphs along the reduction.
- **Strong conv.:** limit inferior of the contexts along the reduction.

### Context

Obtained by

- relabelling the root node of the redex with ⊥, and
- removing all nodes that become unreachable.

---

Partial Order Convergence on Term Graphs

### Convergence

- **Weak conv.:** limit inferior of the term graphs along the reduction.
- **Strong conv.:** limit inferior of the contexts along the reduction.

### Context

Obtained by

- relabelling the root node of the redex with ⊥, and
- removing all nodes that become unreachable.
Partial Order Convergence on Term Graphs

Convergence
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context
Obtained by
- relabelling the root node of the redex with ⊥, and
- removing all nodes that become unreachable.

Example
Partial Order Convergence on Term Graphs

**Convergence**
- Weak conv.: *limit inferior* of the *term graphs* along the reduction.
- Strong conv.: *limit inferior* of the *contexts* along the reduction.

**Context**
Obtained by
- relabelling the *root node* of the redex with $\bot$, and
- removing all nodes that become *unreachable*.

**Example**
```
f
 /\  
f /   \f
|    |
f
|    |
V C   V C
```

Partial Order Convergence on Term Graphs

Convergence
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context
Obtained by
- relabelling the root node of the redex with ⊥, and
- removing all nodes that become unreachable.

Example
![Diagram of a context example]
Partial Order Convergence on Term Graphs

**Convergence**
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

**Context**
Obtained by
- relabelling the root node of the redex with $\perp$, and
- removing all nodes that become unreachable.

**Example**

![Diagram showing the process of context modification and convergence](image)
Partial Order Convergence on Term Graphs

Convergence
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context
Obtained by
- relabelling the root node of the redex with \( \perp \), and
- removing all nodes that become unreachable.

Example

```
\[
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{C} \\
\end{array}
\quad \quad \quad \quad \quad
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{C} \\
\end{array}
\quad \text{context} \quad
\begin{array}{c}
\text{f} \\
\downarrow \\
\perp \\
\downarrow \\
\text{C} \\
\end{array}
\] 
```
**Partial Order Convergence on Term Graphs**

**Convergence**
- Weak conv.: **limit inferior** of the **term graphs** along the reduction.
- Strong conv.: **limit inferior** of the **contexts** along the reduction.

**Context**
Obtained by
- relabelling the **root node** of the redex with \( \bot \), and
- removing all nodes that become **unreachable**.

**Example**

\[
\begin{align*}
\text{context} & \quad \downarrow \quad \text{context} \\
\begin{array}{c}
\begin{array}{c}
\text{f}
\end{array}
\end{array} & \quad \downarrow \quad \begin{array}{c}
\begin{array}{c}
\text{f}
\end{array}
\end{array}
\end{align*}
\]
Partial Order Convergence on Term Graphs

Convergence
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context
Obtained by
- relabelling the root node of the redex with ⊥, and
- removing all nodes that become unreachable.

Example

```
\begin{array}{c}
\text{context} \\
\end{array}
```

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$f$};
\node (b) at (1,0) {$f$};
\node (c) at (0,-1) {$c$};
\node (d) at (1,-1) {$c$};
\node (e) at (0,1) {$f$};
\node (f) at (1,1) {$f$};
\node (g) at (1.5,0) {$\perp$};
\draw[->] (a) to (b);
\draw[->] (b) to (d);
\draw[->] (c) to (a);
\draw[->] (d) to (c);
\draw[->] (e) to (f);
\draw[->] (f) to (e);
\end{tikzpicture}
\end{center}
Metric vs. Partial Order Approach

Recall the situation on terms

For every reduction $S$ in a TRS

$$S : s \xrightarrow{p} t \text{ total} \iff S : s \xrightarrow{m} t.$$

Theorem (soundness of partial order convergence)

For every left-linear, left-finite GRS $R$ we have

$$g \xrightarrow{h} p \cup (\cdot) \cup (R) = R \xrightarrow{t} p \cup (\cdot).$$
### Metric vs. Partial Order Approach

#### Recall the situation on terms

For every reduction \( S \) in a **TRS**

\[
S : s \xrightarrow{p} t \text{ total} \iff S : s \xrightarrow{m} t.
\]

#### On term graphs

For every reduction \( S \) in a **GRS**

\[
S : g \xrightarrow{p} h \text{ total} \iff S : g \xrightarrow{m} h.
\]
Metric vs. Partial Order Approach

Recall the situation on terms
For every reduction $S$ in a TRS
$S: s \xrightarrow{p} t \text{ total } \iff \enspace S: s \xrightarrow{m} t.$

On term graphs
For every reduction $S$ in a GRS
$S: g \xrightarrow{p} h \text{ total } \iff \enspace S: g \xrightarrow{m} h.$

Theorem (soundness of partial order convergence)
For every left-linear, left-finite GRS $\mathcal{R}$ we have
\[
\begin{array}{cccccc}
\mathcal{R} & g & \xrightarrow{p} & h \\
\mathcal{U}(\cdot) & s & \xrightarrow{p} & t
\end{array}
\]
Completeness for Partial Order Convergence

**Theorem (Infinitary normalisation)**

For each term graph $g$, there is a reduction $g \xrightarrow{\rho} h$ to a normal form $h$. 
Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)

For each term graph $g$, there is a reduction $g \xrightarrow{p} h$ to a normal form $h$.

Theorem (Completeness)

Strong $p$-convergence in an orthogonal, left-finite GRS $R$ is complete w.r.t. strong $p$-convergence in $\mathcal{U}(R)$.

\[
\begin{align*}
\mathcal{U}(R) & \quad s \quad \xrightarrow{\cdot} \quad t \quad \xrightarrow{\cdot} \quad t' \\
\mathcal{U}(\cdot) & \quad \uparrow \\
\mathcal{U}(\cdot) & \quad \uparrow \\
R & \quad g \quad \xrightarrow{\cdot} \quad h
\end{align*}
\]
Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)
For each term graph $g$, there is a reduction $g \overset{\mathcal{R}}{\longrightarrow} h$ to a normal form $h$.

Theorem (Completeness)
Strong $p$-convergence in an orthogonal, left-finite GRS $\mathcal{R}$ is complete w.r.t. strong $p$-convergence in $\mathcal{U}(\mathcal{R})$.

Proof.

\[
\begin{array}{c}
\mathcal{U}(\mathcal{R}) \\
\mathcal{U}() \\
\mathcal{R} \\
g
\end{array} \quad \overset{s}{\longrightarrow} \quad \begin{array}{c}
t \\
\end{array}
\]
Completeness for Partial Order Convergence

**Theorem (Infinitary normalisation)**

For each term graph \( g \), there is a reduction \( g \xrightarrow{p} h \) to a normal form \( h \).

**Theorem (Completeness)**

Strong\( p \)-convergence in an orthogonal, left-finite GRS \( \mathcal{R} \) is complete w.r.t. strong\( p \)-convergence in \( \mathcal{U}(\mathcal{R}) \).

**Proof.**

\[
\begin{array}{c}
\mathcal{U}(\mathcal{R}) \\
\mathcal{U}(\cdot) \\
\downarrow \\
\mathcal{R} \\
g
\end{array} \xrightarrow{s} \\
\begin{array}{c}
\begin{array}{c}
\mathcal{U}(\cdot) \\
\downarrow \\
\mathcal{R} \\
g
\end{array} \\
t \\
h
\end{array} \xrightarrow{\text{normalising}}
\]
Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)

For each term graph $g$, there is a reduction $g \Rightarrow p h$ to a normal form $h$.

Theorem (Completeness)

Strong $p$-convergence in an orthogonal, left-finite GRS $\mathcal{R}$ is complete w.r.t. strong $p$-convergence in $\mathcal{U}(\mathcal{R})$.

Proof.

\[ \mathcal{U}(\mathcal{R}) \quad s \rightarrow t \quad \text{soundness} \]

\[ \mathcal{U}(\cdot) \]

\[ \mathcal{R} \quad g \rightarrow h \quad \text{normalising} \]

\[ \mathcal{U}(\cdot) \]

\[ t' \]

\[ \mathcal{U}(\cdot) \]
Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)
For each term graph \( g \), there is a reduction \( g \rightarrow h \) to a normal form \( h \).

Theorem (Completeness)

Strong \( p \)-convergence in an orthogonal, left-finite GRS \( \mathcal{R} \) is complete w.r.t. strong \( p \)-convergence in \( \mathcal{U}(\mathcal{R}) \).

Proof.

\[
\begin{align*}
\overline{\mathcal{U}(\mathcal{R})} & \xrightarrow{s} t \\
\overline{\mathcal{U}(\cdot)} & \xrightarrow{t'} h \\
\mathcal{R} & \xrightarrow{g} t
\end{align*}
\]
Conclusions

<table>
<thead>
<tr>
<th>Infinitary term graph rewriting</th>
</tr>
</thead>
<tbody>
<tr>
<td>intuitive &amp; simple generalisation</td>
</tr>
<tr>
<td>however: weak convergence is wacky</td>
</tr>
<tr>
<td>strong convergence is well-behaved</td>
</tr>
</tbody>
</table>
Conclusions

Infinitary term graph rewriting

- intuitive & simple generalisation
- however: weak convergence is wacky
- strong convergence is well-behaved

Is it relevant?

- connection to lazy functional programming
- soundness & completeness
Conclusions

Infinitary term graph rewriting

- intuitive & simple generalisation
- however: weak convergence is wacky
- strong convergence is well-behaved

Is it relevant?

- connection to lazy functional programming
- soundness & completeness

Completeness of $m$-convergence for normalising reductions

\[
\begin{align*}
\mathcal{U}(\mathcal{R}) &\quad s \\
\mathcal{U}(\cdot) &\quad \uparrow \\
\mathcal{R} &\quad \mathfrak{g} \\
\mathcal{U}(\cdot) &\quad \uparrow \\
t &\quad \mathfrak{h}
\end{align*}
\]