Soundness and Completeness of Infinitary Term Graph Rewriting

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From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]
From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]
From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]
From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]
From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]

\[ a \rightarrow b \]
From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]

unravel

\[ a \rightarrow b \]
From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]

\[ a \rightarrow b \]
From Terms to Term Graphs

\[ f(\, g(a), h(\, g(a), a) \, ) \]
From Terms to Term Graphs

\[ f(g(a)), h(g(a), a) \]

unravel
From Terms to Term Graphs

\[ f(g(a)), h(g(a), a) \]
From Terms to Term Graphs

```
f(g(a), h(g(a), a))
```

```
unravel

b → c
```
From Terms to Term Graphs

\[ f(g(a), h(g(a), a)) \]

unravel

\[ b \rightarrow c \]
Infinitary Term Graph Rewriting – What is it for?

A common formalism

study correspondences between infinitary term rewriting and finitary term graph rewriting
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A common formalism

study correspondences between infinitary term rewriting and finitary term graph rewriting

Lazy evaluation

- infinitary term rewriting only covers non-strictness
- however: lazy evaluation = non-strictness + sharing
Infinitary Term Graph Rewriting – What is it for?

A common formalism
study correspondences between infinitary term rewriting and finitary term graph rewriting

Lazy evaluation
- infinitary term rewriting only covers non-strictness
- however: lazy evaluation = non-strictness + sharing

towards infinitary lambda calculi with letrec
- Ariola & Blom. Skew confluence and the lambda calculus with letrec.
- the calculus is non-confluent
- but there is a notion of infinite normal forms
Obstacles

What is the an appropriate notion of convergence on term graph?

- generalise convergence on terms
  - But: there are many quite different generalisations.
  - Most important issue: how to deal with sharing?
- simulate infinitary term rewriting in a **sound & complete** manner
Obstacles

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Completeness w.r.t. term graph rewriting

An issue even for finitary acyclic term graph reduction!
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An issue even for finitary acyclic term graph reduction!

\[ s \xrightarrow{\mathcal{U} (\cdot)}^* t \]
Obstacles

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Completeness w.r.t. term graph rewriting

An issue even for finitary acyclic term graph reduction!

\[
\begin{array}{c}
\overset{s}{U}(\cdot) \quad \overset{\cdot}{U}(\cdot) \\
\downarrow \quad \downarrow
\end{array}
\quad \overset{*}{\rightarrow}
\begin{array}{c}
t \\
h
\end{array}
\]

\[
\begin{array}{c}
g \\
\end{array}
\quad \overset{\cdot}{U}(\cdot) \quad \overset{*}{\rightarrow}
\begin{array}{c}
h
\end{array}
\]
Obstacles

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Completeness w.r.t. term graph rewriting

An issue even for finitary acyclic term graph reduction!

\[
\begin{align*}
  s & \xrightarrow{\ast} t & \xrightarrow{\ast} & t' \\
  g & \quad \uparrow \mathcal{U} (\cdot) \quad \uparrow \mathcal{U} (\cdot) \\
  & \quad g & \xrightarrow{\ast} & h
\end{align*}
\]
Obstacles

What is the an appropriate notion of convergence on term graph?

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Completeness w.r.t. term graph rewriting

An issue even for finitary acyclic term graph reduction!

\[ s \xrightarrow{U(\cdot)} t \xrightarrow{^*} t' \]
\[ g \xrightarrow{U(\cdot)} h \xrightarrow{^*} \]
Outline

1. Introduction
   - Goals
   - Obstacles

2. Modes of Convergence on Term Graphs
   - Metric Approach
   - Partial Order Approach
   - Metric vs. Partial Order Approach
Metric Infinitary Term Rewriting

Complete metric on terms

- terms are endowed with a complete metric in order to formalise the convergence of infinite reductions.
- metric distance between terms:

\[ d(s, t) = 2^{-\text{sim}(s, t)} \]

\[ \text{sim}(s, t) = \text{minimum depth } d \text{ s.t. } s \text{ and } t \text{ differ at depth } d \]
Complete metric on terms

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Strong convergence via metric \( d \) and redex depth

- convergence in the metric space \( (\mathcal{T}^\infty(\Sigma), d) \)
  \[ \rightsquigarrow \text{ depth of the differences between the terms has to tend to infinity} \]
- depth of redexes has to tend to infinity
Example: Strongly Converging

\[ f \downarrow a \rightarrow g(a) \]
Example: Strongly Converging

\[ a \rightarrow g(a) \]
Example: Strongly Converging

\[ a \rightarrow g(a) \]
Example: Strongly Converging

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A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.
A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.

Truncation of term graphs

The truncation $g_{\dag d}$ is obtained from $g$ by

- relabelling all nodes at depth $d$ with $\bot$, and
- removing all nodes that thus become unreachable from the root.

**Simple metric on term graphs**

$d_{\dag}(g, h) = 2 - \text{sim}_{\dag}(g, h)$

Where $\text{sim}_{\dag}(g, h) = \text{maximum depth } d \text{ s.t. } g_{\dag d} \sim = h_{\dag d}$.

Strong convergence via metric $d_{\dag}$ and redex depth convergence in the metric space $(G_{\infty}C(\Sigma), d_{\dag})$ depth of redexes has to tend to infinity.
A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.

Truncation of term graphs

The truncation $g^{\uparrow d}$ is obtained from $g$ by

- relabelling all nodes at depth $d$ with $\perp$, and
- removing all nodes that thus become unreachable from the root.

The simple metric on term graphs

$$d^\uparrow(g, h) = 2^{-\text{sim}^\uparrow(g, h)}$$

Where $\text{sim}^\uparrow(g, h) = \text{maximum depth } d \text{ s.t. } g^{\uparrow d} \simeq h^{\uparrow d}$. 
A Metric on Term Graphs

Depth of a node = length of a shortest path from the root to the node.

Truncation of term graphs

The truncation $g \uparrow d$ is obtained from $g$ by

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The simple metric on term graphs

$$d_\uparrow(g, h) = 2^{-\text{sim}_\uparrow(g, h)}$$

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Strong convergence via metric $d_\uparrow$ and redex depth

- convergence in the metric space $(G_\infty^c(\Sigma), d_\uparrow)$
- depth of redexes has to tend to infinity
Soundness & Completeness

Theorem (soundness of metric convergence)

For every left-linear, left-finite GRS $R$ we have $g \rightarrow m R h \Rightarrow U(g) \rightarrow m U(R) U(h)$. 

Completeness property
Soundness & Completeness

Theorem (soundness of metric convergence)

For every left-linear, left-finite GRS \( \mathcal{R} \) we have

\[
g \xrightarrow{m_R} h \quad \implies \quad \mathcal{U}(g) \xrightarrow{m_{\mathcal{U}(\mathcal{R})}} \mathcal{U}(h).
\]
Soundness & Completeness

Theorem (soundness of metric convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

$$g \xrightarrow{m}_{\mathcal{R}} h \quad \implies \quad \mathcal{U}(g) \xrightarrow{m}_{\mathcal{U}(\mathcal{R})} \mathcal{U}(h).$$

Completeness property

\[
\begin{array}{ccc}
\mathcal{U}(\mathcal{R}) & \xrightarrow{s} & t \\
\mathcal{U}(\cdot) & & \\
\mathcal{R} & \xrightarrow{g} & \\
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Soundness & Completeness

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For every left-linear, left-finite GRS \( \mathcal{R} \) we have

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t' \\\\cdot\end{array} \xrightarrow{t'} \begin{array}{c}
\mathcal{U}(\cdot) \\
h
\end{array}
\]
Completeness of Infinitary Term Graph Rewriting?

We have a rule $n(x, y) \rightarrow n + 1(x, y)$ for each $n \in \mathbb{N}$.

[Kennaway et al., 1994]
Completeness of Infinitary Term Graph Rewriting?

We have a rule \( n(x, y) \rightarrow n + 1(x, y) \) for each \( n \in \mathbb{N} \).

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   • Obstacles

2 Modes of Convergence on Term Graphs
   • Metric Approach
   • Partial Order Approach
   • Metric vs. Partial Order Approach
Partial Order Infinitary Term Rewriting

Partial order on terms

- **partial terms**: terms with additional constant \( \bot \) (read as “undefined”)
- partial order \( \leq \bot \) reads as: “is less defined than”
- \( \leq \bot \) is a complete semilattice (\( = \text{cpo} + \text{glbs of non-empty sets} \))
Partial Order Infinitary Term Rewriting

Partial order on terms

- **partial terms**: terms with additional constant $\perp$ (read as “undefined”)
- partial order $\leq_{\perp}$ reads as: “is less defined than”
- $\leq_{\perp}$ is a complete semilattice (=$\text{cpo} + \text{glbs of non-empty sets}$)

Convergence

- formalised by the limit inferior:

$$\liminf_{l \to \alpha} t_l = \bigsqcup_{\beta < \alpha} \bigcap_{\beta \leq l < \alpha} t_l$$

- intuition: eventual persistence of nodes of the terms
- **weak convergence**: limit inferior of the terms of the reduction
Partial Order Infinitary Term Rewriting

Partial order on terms
- **Partial terms**: terms with additional constant \( \perp \) (read as “undefined”)
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Convergence
- formalised by the limit inferior:
  \[
  \liminf_{\iota \to \alpha} t_{\iota} = \bigsqcup_{\beta < \alpha} \bigcap_{\beta \leq \iota < \alpha} t_{\iota}
  \]
- intuition: eventual persistence of nodes of the terms
- weak convergence: limit inferior of the terms of the reduction
- strong convergence: limit inferior of the contexts of the reduction
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Partial Order Convergence vs. Metric Convergence

Intuition of partial order convergence

- subterms that would break $m$-convergence, converge to $\bot$
- every (continuous) reduction converges

Theorem (total $p$-convergence = $m$-convergence)

For every reduction $S$ in a TRS the following equivalence holds:

Theorem (normalisation & confluence)

Every orthogonal TRS is infinitarily normalising and infinitarily confluent w.r.t. strong $p$-convergence.
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\[ S: s \xrightarrow{p} t \text{ total} \quad \text{iff} \quad S: s \xrightarrow{m} t \]
\section*{Partial Order Convergence vs. Metric Convergence}

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      \item every (continuous) reduction converges
    \end{itemize}
\end{itemize}

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A Partial Order on Term Graphs – How?

Specialise on terms

- Consider terms as **term trees** (i.e. term graphs with tree structure)
- How to define the partial order $\leq_\bot$ on term trees?

- $\bot$-homomorphisms $\varphi : \mathbf{g} \rightarrow \bot\mathbf{h}$
- Homomorphism condition suspended on $\bot$-nodes
- Allow mapping of $\bot$-nodes to arbitrary nodes

**Proposition** (\$\bot\$-homomorphisms characterise $\leq_\bot$ on terms)

For all $s, t \in T_\infty(\Sigma_\bot)$: $s \leq_\bot t$ iff $\exists \varphi : s \rightarrow \bot t$. 

**Definition** (Simple partial order $\leq_\bot$ on term graphs)

For all $g, h \in G_\infty(\Sigma_\bot)$, let $g \leq_\bot h$ iff there is some $\varphi : g \rightarrow \bot h$. 

### A Partial Order on Term Graphs – How?

**Specialise on terms**
- Consider terms as **term trees** (i.e. term graphs with tree structure)
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**$\bot$-homomorphisms $\phi: g \rightarrow_{\bot} h$**
- Homomorphism condition suspended on $\bot$-nodes
- Allow mapping of $\bot$-nodes to arbitrary nodes
- Same mechanism that formalises matching in term graph rewriting

---

**Proposition** (Homomorphisms characterise $\leq_{\bot}$ on terms)

For all $s, t \in T_\infty(\Sigma_{\bot})$: $s \leq_{\bot} t$ iff $\exists \phi: s \rightarrow_{\bot} t$.
A Partial Order on Term Graphs – How?

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### A Partial Order on Term Graphs – How?

**Specialise on terms**
- Consider terms as **term trees** (i.e. term graphs with tree structure)
- How to define the partial order $\leq_\bot$ on term trees?

**$\bot$-homomorphisms $\phi : g \rightarrow_\bot h$**
- homomorphism condition suspended on $\bot$-nodes
- allow mapping of $\bot$-nodes to arbitrary nodes
- same mechanism that formalises matching in term graph rewriting

**Proposition ($\bot$-homomorphisms characterise $\leq_\bot$ on terms)**

For all $s, t \in T^\infty(\Sigma_\bot)$: $s \leq_\bot t$ iff $\exists \phi : s \rightarrow_\bot t$

**Definition (Simple partial order $\leq^{S_\bot}$ on term graphs)**

For all $g, h \in G^\infty(\Sigma_\bot)$, let $g \leq^{S_\bot} h$ iff there is some $\phi : g \rightarrow_\bot h$. 
Partial Order Convergence on Term Graphs

Convergence

- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.
Partial Order Convergence on Term Graphs

**Convergence**

- Weak conv.: limit inferior of the **term graphs** along the reduction.
- Strong conv.: limit inferior of the **contexts** along the reduction.

**Context**

Term graph obtained by relabelling the root node of the redex with $\bot$ (and removing all nodes that become unreachable).
Partial Order Convergence on Term Graphs

Convergence
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context
Term graph obtained by relabelling the root node of the redex with ⊥ (and removing all nodes that become unreachable).

Example

```
  f
 /\  \\
 f | f
 |  \  |
 v   v
 C   C
```

Partial Order Convergence on Term Graphs

Convergence
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context
Term graph obtained by relabelling the root node of the redex with $\bot$ (and removing all nodes that become unreachable).

Example
![Diagram of term graphs with relabeling and removal of unreachable nodes]
Partial Order Convergence on Term Graphs

Convergence

- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context

Term graph obtained by relabelling the root node of the redex with \( ot \) (and removing all nodes that become unreachable).

Example

```
  f  
 / \  
f  f  
|   |  
c  c  
```

context

---

\( f \) \( f \) \( c \) \( f \) \( c \)
Partial Order Convergence on Term Graphs

Convergence

- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context

Term graph obtained by relabelling the root node of the redex with ⊥ (and removing all nodes that become unreachable).

Example

```
           f
          / \  
         /   \ 
        f----f
          \  /  
           C C
```

```
           f
          /   
         /     
        f-----f
          \  /  
           C C
```

```
           f
          /   
         /     
        f-----f
          \  /  
           C C
```
Partial Order Convergence on Term Graphs

**Convergence**
- Weak conv.: limit inferior of the term graphs along the reduction.
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**Example**

![Example Diagram](image)
Partial Order Convergence on Term Graphs

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- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

**Context**
Term graph obtained by relabelling the root node of the redex with $\perp$ (and removing all nodes that become unreachable).

**Example**

```
  f
 / \  \
/   \ 
C     C
```

context

```
  f
 /  \ 
/    \ 
C     C
```

```
  f
 /  \  \
/    \  
C     C
```
Partial Order Convergence on Term Graphs

Convergence
- Weak conv.: limit inferior of the term graphs along the reduction.
- Strong conv.: limit inferior of the contexts along the reduction.

Context
Term graph obtained by relabelling the root node of the redex with \( \perp \) (and removing all nodes that become unreachable).

Example

\[ \begin{array}{c}
\text{context}
\end{array} \]

\[
\begin{array}{c}
\text{f} \\
\text{f} \\
\text{f} \\
\text{C} \\
\text{C}
\end{array}
\rightarrow
\begin{array}{c}
\text{f} \\
\text{f} \\
\text{f} \\
\text{C} \\
\perp
\end{array}
\]
Recall the situation on terms
For every reduction $S$ in a TRS

\[ S: s \xrightarrow{P} t \text{ total} \iff \quad S: s \xrightarrow{m} t. \]
Metric vs. Partial Order Approach

Recall the situation on terms
For every reduction $S$ in a TRS

$S : s \xrightarrow{p} t$ total $\iff S : s \xrightarrow{m} t.$

On term graphs
For every reduction $S$ in a GRS

$S : s \xrightarrow{p} t$ total $\iff S : s \xrightarrow{m} t.$
Recall the situation on terms

For every reduction $S$ in a TRS

$$S: s \xrightarrow{P} t \text{ total } \iff S: s \xrightarrow{m} t.$$  

On term graphs

For every reduction $S$ in a GRS

$$S: s \xrightarrow{P} t \text{ total } \iff S: s \xrightarrow{m} t.$$  

Theorem (soundness of partial order convergence)

For every left-linear, left-finite GRS $R$ we have

$$g \xrightarrow{P} R h \implies \mathcal{U}(g) \xrightarrow{P} \mathcal{U}(R) \mathcal{U}(h).$$
Failure of Completeness for Metric Convergence

We have a rule \( n(x, y) \rightarrow n + 1(x, y) \) for each \( n \in \mathbb{N} \).
Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)

For each term graph \( g \), there is a reduction \( g \overset{p}{\rightarrow} h \) to a normal form \( h \).
Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)

For each term graph \( g \), there is a reduction \( g \xrightarrow{p} h \) to a normal form \( h \).

Theorem (Completeness)

Strong \( p \)-convergence in an orthogonal, left-finite GRS \( \mathcal{R} \) is complete w.r.t. strong \( p \)-convergence in \( \mathcal{U}(\mathcal{R}) \).

\[
\begin{array}{ccccccc}
\mathcal{U}(\mathcal{R}) & s & \to & t & \to & t' \\
\mathcal{U}(\cdot) & \uparrow & & & & \downarrow \\
\mathcal{R} & g & \to & h \\
\end{array}
\]
Completeness for Partial Order Convergence

Theorem (Infinitary normalisation)
For each term graph \( g \), there is a reduction \( g \not\rightarrow^* h \) to a normal form \( h \).

Theorem (Completeness)
Strong \( p \)-convergence in an orthogonal, left-finite GRS \( \mathcal{R} \) is complete w.r.t. strong \( p \)-convergence in \( \mathcal{U}(\mathcal{R}) \).

Proof.

\[
\begin{array}{c}
\mathcal{U}(\mathcal{R}) \\
\downarrow \mathcal{U}(\cdot) \\
\mathcal{R} \quad g \\
\end{array} \quad \xrightarrow{s} \quad \begin{array}{c}
\mathcal{U}(\mathcal{R}) \\
\downarrow \mathcal{U}(\cdot) \\
\mathcal{R} \quad t \\
\end{array}
\]
Completeness for Partial Order Convergence

**Theorem (Infinitary normalisation)**

*For each term graph* \( g \), *there is a reduction* \( g \overset{p}{\rightarrow} h \) *to a normal form* \( h \).

**Theorem (Completeness)**

*Strong* \( p \)-*convergence in an orthogonal, left-finite GRS* \( R \) *is complete w.r.t. strong* \( p \)-*convergence in* \( U(R) \).

**Proof.**

\[
\begin{array}{ccccccc}
U(R) & s & \rightarrow & t \\
U(\cdot) & U(\cdot) & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
R & g & \rightarrow & h \\
\end{array}
\]

normalising
Completeness for Partial Order Convergence

**Theorem (Infinitary normalisation)**

For each term graph $g$, there is a reduction $g \overset{p}{\Rightarrow} h$ to a normal form $h$.

**Theorem (Completeness)**

Strong $p$-convergence in an orthogonal, left-finite GRS $\mathcal{R}$ is complete w.r.t. strong $p$-convergence in $\mathcal{U}(\mathcal{R})$.

**Proof.**

Diagram showing the soundness and normalising transitions between $\mathcal{U}(\mathcal{R})$, $\mathcal{U}(\cdot)$, $\mathcal{R}$, $g$, $s$, $t$, $t'$, $\mathcal{U}(\cdot)$, and $h$. The transitions are labeled as soundness and normalising.
Completeness for Partial Order Convergence

**Theorem (Infinitary normalisation)**
For each term graph $g$, there is a reduction $g \xrightarrow{p} h$ to a normal form $h$.

**Theorem (Completeness)**
Strong $p$-convergence in an orthogonal, left-finite GRS $\mathcal{R}$ is complete w.r.t. strong $p$-convergence in $\mathcal{U}(\mathcal{R})$.

**Proof.**

\[
\begin{align*}
\mathcal{U}(\mathcal{R}) & \xrightarrow{s} t \\
\mathcal{U}(\cdot) & \\
\mathcal{R} & \xrightarrow{g} h
\end{align*}
\]
Conclusions

Infinitary term graph rewriting

- intuitive & simple generalisation
- however: weak convergence is wacky
- strong convergence is well-behaved
## Conclusions

**Infinitary term graph rewriting**
- Intuitive & simple generalisation
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**Is it relevant?**
- Connection to lazy functional programming
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Conclusions

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Is it relevant?
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Completeness of $m$-convergence for normalising reductions

\[
\begin{align*}
\mathcal{U}(R) &\quad \mathcal{U}(\cdot) \uparrow \\
\mathcal{U}(\cdot) \quad \mathcal{U}(\cdot) &\quad \mathcal{U}(\cdot) \quad \mathcal{U}(\cdot)
\end{align*}
\]