Convergence in Infinitary Term Graph Rewriting Systems is Simple

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Term Graph Rewriting vs. Infinitary Rewriting

Pick one to avoid the other.
Term Graph Rewriting vs. Infinitary Rewriting

Pick one to avoid the other.

Pick **term graph rewriting**
- finite representation of infinite terms (via cycles)
- finite representation of infinite rewrite sequences

\[
\begin{align*}
\text{f} &\quad \text{g} &\quad \text{h} \\
b &\quad &\quad \\
\end{align*}
\]
Term Graph Rewriting vs. Infinitary Rewriting

Pick one to avoid the other.

**Pick term graph rewriting**
- finite representation of infinite terms (via cycles)
- finite representation of infinite rewrite sequences

**Pick infinitary rewriting**
- avoid dealing with term graphs
- work on the unravelling instead

![Diagram](image)
A common formalism

study *correspondences* between infinitary TRSs and finitary GRSs
Infinitary Term Graph Rewriting – What is it for?

A common formalism
study correspondences between infinitary TRSs and finitary GRSs

Lazy evaluation
- infinitary term rewriting only covers non-strictness
- however: lazy evaluation = non-strictness + sharing
Infinitary Term Graph Rewriting – What is it for?

A common formalism
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Lazy evaluation
- infinitary term rewriting *only covers non-strictness*
- however: lazy evaluation = non-strictness + sharing

Towards infinitary lambda calculi with letrec
- Ariola & Blom. *Skew confluence and the lambda calculus with letrec.*
- the calculus is *non-confluent*
- but there is a notion of *infinite normal forms*
**Our Previous Approach [RTA ’11]**

### Profile

- weak convergence
- two modes of convergence: metric & partial order
## Our Previous Approach [RTA ’11]

**Profile**

- weak convergence
- two modes of convergence: metric & partial order
- result:  
  - correspondence between metric & partial order convergence  
  - soundness w.r.t. infinitary term rewriting (sorta kinda)
Our Previous Approach

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metric convergence ⇔ partial order convergence “without ⊥’s”
Our Previous Approach [RTA '11]

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- weak convergence
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  soundness w.r.t. infinitary term rewriting (sorta kinda)

convergence and unravelling commute:

\[
\begin{align*}
S(G) \xrightarrow{\lim} G & \\
U \Downarrow & \\
S(T) \xrightarrow{\lim} T
\end{align*}
\]
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- result:  ▶ correspondence between metric & partial order convergence
  ▶ soundness w.r.t. infinitary term rewriting (sorta kinda)
- problem: complicated; difficult to analyse; lack of completeness
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Term graph rewriting with \( \text{from}(x) \rightarrow x :: \text{from}(s(x)) \)

from
  ↓
0
Our Previous Approach [RTA ’11]

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Term graph rewriting with \( \text{from}(x) \rightarrow x :: \text{from}(s(x)) \)

```
from
\downarrow
0
\rightarrow
::
\downarrow
\downarrow
0
from
\downarrow
s
```
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Term graph rewriting with \( \text{from}(x) \to x::\text{from}(s(x)) \)
Our New Approach

<table>
<thead>
<tr>
<th>Less restrictive structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_R(g, h) \geq d_S(g, h)$</td>
</tr>
</tbody>
</table>
Our New Approach

Less restrictive structures

\[ d_R(g, h) \geq d_S(g, h) \]

old

new
Our New Approach

Less restrictive structures

\[ d_R(g, h) \geq d_S(g, h) \]

\[ \leadsto \text{coarser topology (i.e. more sequences converge)} \]
Our New Approach

Less restrictive structures

- \( d_R(g, h) \geq d_S(g, h) \)
  ~\( \Rightarrow \) coarser topology (i.e. more sequences converge)

- \( g \leq^R h \implies g \leq^S h \)
Our New Approach

Less restrictive structures

- \( d_R(g, h) \geq d_S(g, h) \)
  \( \Rightarrow \) coarser topology (i.e. more sequences converge)

- \( g \leq_R h \implies g \leq_S h \)
  \( \Rightarrow \) sequences converge to term graphs “with fewer \( \perp \)’s”
Our New Approach

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Term graph rewriting with \( \text{from}(x) \rightarrow x :: \text{from}(s(x)) \)
Outline

1. Introduction
   - Goals
   - A Different Approach

2. Weak Convergence

3. Strong Convergence
Metric Infinitary Term Rewriting

Complete metric on terms

\[ d(s, t) = 2^{-\text{sim}(s, t)} \]

\[ \text{sim}(s, t) = \text{maximum depth } d \text{ s.t. truncated at depth } d, \text{ s and t are equal} \]
Metric Infinitary Term Rewriting

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Example

\[ f \]
\[ a \quad f \]
\[ b \quad c \]
\[ \underline{s} \]

\[ f \]
\[ a \quad e \]
\[ a \]
\[ \underline{t} \]
Metric Infinitary Term Rewriting

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Example

\[
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{b} \\
\downarrow \\
\text{s}
\end{array}
\quad
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{t}
\end{array}
\]

\[
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{f} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{e} \\
\downarrow \\
\text{c}
\end{array}
\quad
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{a}
\end{array}
\]
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Example

\[ f \]
\[ a \]
\[ b \]
\[ s \]
\[ c \]
\[ t \]
\[ a \]
\[ e \]
\[ f \]
\[ a \]
Metric Infinitary Term Rewriting

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Example

```
  f
  / \    \\
 a   f   a   e
  |   |   |   |
 b   c   e   a
 s   t
```

1 level
Metric Infinitary Term Rewriting

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Example

\[ d(\underbrace{s}_{1 \text{ level}}, \underbrace{t}_{1 \text{ level}}) = 2^{-1} \]
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Example

\[
\begin{array}{c}
\text{1 level} \\
\hline
\text{f} \quad \text{f} \\
\downarrow \quad \downarrow \\
\text{a} \quad \text{a} \\
\downarrow \quad \downarrow \\
\text{b} \quad \text{c} \\
\downarrow \quad \downarrow \\
\text{f} \quad \text{f} \\
\downarrow \quad \downarrow \\
\text{a} \quad \text{e} \\
\end{array}
\]

\[ d(s, t) = 2^{-1} \]

\[
\begin{array}{c}
\text{2 levels} \\
\hline
\text{f} \\
\downarrow \\
\text{a} \\
\downarrow \\
\text{a} \\
\end{array}
\]

\[
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a} \\
\end{array}
\]

\[
\begin{array}{c}
\text{f} \\
\downarrow \\
\text{a} \\
\end{array}
\]

\[ d(s', t') = 2^{-1} \]
Metric Infinitary Term Rewriting

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Example

\[ d(s, t) = 2^{-1} \]

\[ d(s', t') = 2^{-2} \]
Complete metric on terms

\[ d(g, h) = 2^{-\text{sim}(g, h)} \]

\[ \text{sim}(g, h) = \text{maximum depth } d \text{ s.t. truncated at depth } d, \ g \text{ and } h \text{ are equal} \]

Example

\[ d(g, h) = 2^{-1} \]

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Partial Order Infinitary Term Rewriting

Partial order on terms

- **partial terms**: terms with additional constant ⊥ (read as “undefined”)
- **partial order** ≤⊥ reads as: “is less defined than”
- ≤⊥ is a **complete semilattice** (= cpo + glbs of non-empty sets)
Partial Order Infinitary Term Rewriting

Partial order on terms
- **partial terms**: terms with additional constant ⊥ (read as “undefined”)
- partial order $\leq_\bot$ reads as: “is less defined than”
- $\leq_\bot$ is a complete semilattice (＝ cpo + glbs of non-empty sets)

Convergence
- formalised by the limit inferior:
  \[
  \liminf_{\iota \to \alpha} t_\iota = \bigsqcup_{\beta < \alpha} \bigcap_{\beta \leq \iota < \alpha} t_\iota
  \]
- intuition: eventual persistence of nodes of the terms
A Partial Order on Term Graphs

Specialise on terms

- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order $\leq_{\perp}$ on term trees?
A Partial Order on Term Graphs

Specialise on terms

- Consider terms as **term trees** (i.e. term graphs with tree structure)
- How to define the partial order $\leq_{\perp}$ on term trees?

$\perp$-homomorphisms $\phi: g \rightarrow_{\perp} h$

- homomorphism condition suspended on $\perp$-nodes
- allow mapping of $\perp$-nodes to arbitrary nodes
- same mechanism describing matching in term graph rewriting
A Partial Order on Term Graphs

Specialise on terms
- Consider terms as term trees (i.e. term graphs with tree structure)
- How to define the partial order $\leq_\perp$ on term trees?

$\perp$-homomorphisms $\phi : g \rightarrow_\perp h$
- homomorphism condition suspended on $\perp$-nodes
- allow mapping of $\perp$-nodes to arbitrary nodes
- same mechanism describing matching in term graph rewriting

Definition (Simple partial order $\leq^S_\perp$ on term graphs)
For all $g, h \in \mathcal{G}^\infty(\Sigma_\perp)$, let $g \leq^S_\perp h$ iff there is some $\phi : g \rightarrow_\perp h$. 
Properties of Completions

Term graph rewriting with $\text{from}(x) \to x :: \text{from}(s(x))$

Theorem (metric completion of term graphs)

The metric completion of $(\mathcal{G}_C(\Sigma), \mathcal{d}_S)$ is the metric space $(\mathcal{G}_\infty C(\Sigma), \mathcal{d}_S)$.

Theorem (ideal completion of term graphs)

The ideal completion of $(\mathcal{G}_C(\Sigma \perp), \leq S \perp)$ is order isomorphic to $(\mathcal{G}_\infty C(\Sigma \perp), \leq S \perp)$.
Properties of Completions

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

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The metric completion of $(G_C(\Sigma), d_S)$ is the metric space $(G_C^\infty(\Sigma), d_S)$.
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Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

Theorem (metric completion of term graphs)

The metric completion of $(G_C(\Sigma), d_S)$ is the metric space $(G_C^\infty(\Sigma), d_S)$.

Theorem (ideal completion of term graphs)

The ideal completion of $(G_C(\Sigma_\bot), \leq_S)$ is order isomorphic to $(G_C^\infty(\Sigma_\bot), \leq_S)$.
Metric vs. Partial Order Convergence

Partial order convergence

\[ f \rightarrow f \rightarrow f \rightarrow f \rightarrow f \rightarrow \ldots \]

\[ c \rightarrow c \rightarrow c \rightarrow c \rightarrow c \rightarrow c \]
Metric vs. Partial Order Convergence

Partial order convergence:

\[ f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \cdots \quad f \]

\[ c \quad c \quad c \quad c \quad c \quad c \quad c \quad c \]

Theorem

Let \( S \) be a reduction in a GRS:

\[ S : g, \quad \rightarrow \quad m \quad R \quad h = \Rightarrow \quad \iff \]

Why???
## Metric vs. Partial Order Convergence

### Partial order convergence

<table>
<thead>
<tr>
<th>f</th>
<th>→</th>
<th>f</th>
<th>→</th>
<th>f</th>
<th>→</th>
<th>f</th>
<th>→</th>
<th>f</th>
<th>→</th>
<th>f</th>
<th>→</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>→</td>
<td>C</td>
<td>→</td>
<td>C</td>
<td>→</td>
<td>C</td>
<td>→</td>
<td>C</td>
<td>→</td>
<td>C</td>
<td>→</td>
<td>C</td>
</tr>
</tbody>
</table>

Why???

Theorem

Let $S$ be a reduction in a GRS $R$:

$$S : g, \rightarrow m \Rightarrow h \Rightarrow \Rightarrow S : g, \rightarrow p \Rightarrow h \text{ total}$$
Metric vs. Partial Order Convergence

Partial order convergence

\[ f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad \ldots \quad \rightarrow \quad f \]

\[ \underbrace{c \quad c \quad c \quad c \quad c \quad c} \quad \underbrace{c \quad c} \]

Why???

Because

\[ f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad f \]

\[ \underbrace{c \quad c \quad c} \quad \underbrace{c} \]
Metric vs. Partial Order Convergence

Partial order convergence

\[ f \rightarrow f \rightarrow f \rightarrow f \rightarrow f \rightarrow \ldots \rightarrow f \]

\[ c \rightarrow c \rightarrow c \rightarrow c \rightarrow c \rightarrow c \rightarrow c \]

Why???

Because

\[ f \rightarrow f \leq_{S} \perp \rightarrow f \rightarrow f \]
Metric vs. Partial Order Convergence

Partial order convergence

\[ f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad f \quad \rightarrow \quad \cdots \quad f \]

\[ c \quad c \quad c \quad c \quad c \quad c \quad c \]

Why???

Because

\[ f \quad \prec_{\perp}^{S} \quad f \]

\[ c \quad c \quad c \]

Theorem

Let \( S \) be a reduction in a GRS \( \mathcal{R} \):

\[ S: \ g \overset{m}{\rightarrow}_{\mathcal{R}} h \quad \overset{\checkmark}{\rightarrow} \quad S: \ g \overset{p}{\rightarrow}_{\mathcal{R}} h \text{ total} \]
Metric vs. Partial Order Convergence

Partial order convergence

\[
\begin{array}{cccccc}
  f & \rightarrow & f & \rightarrow & f & \rightarrow & f & \rightarrow & f & \rightarrow & f & \rightarrow & f & \rightarrow & f \\
  c & \downarrow & c & \downarrow & c & \downarrow & c & \downarrow & c & \downarrow & c & \downarrow & c & \downarrow & c
\end{array}
\]

Why???

Because

\[
\begin{array}{c}
  f \\
  c \\
  c
\end{array} \leq_{S} \begin{array}{c}
  f \\
  c
\end{array}
\]

Theorem

Let \( S \) be a reduction in a GRS \( \mathcal{R} \):

\[
S: \ g \xrightarrow{m_{\mathcal{R}}} h \quad \checkmark \quad S: \ g \xrightarrow{p_{\mathcal{R}}} h \text{ total}
\]
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Strong Convergence

Intuition behind strong convergence

- syntactic restriction of convergence
- pretend that the root of the left-hand side and the right-hand side of each rule are distinct
Strong Convergence

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Strong metric convergence
additional restriction: depth of contracted redexes must tend to infinity
Strong Convergence

Intuition behind strong convergence

- syntactic restriction of convergence
- pretend that the root of the left-hand side and the right-hand side of each rule are distinct

Strong metric convergence

additional restriction: depth of contracted redexes must tend to infinity

Strong partial order convergence

modify limit formation: replace each redex with ⊥
Consequences

Partial order convergence

\[ f \rightarrow f \rightarrow f \rightarrow f \rightarrow \ldots \]

\[ C \rightarrow C \rightarrow C \rightarrow C \rightarrow C \rightarrow C \rightarrow C \rightarrow \ldots \]
Consequences

Partial order convergence

\[ f \rightarrow f \rightarrow f \rightarrow f \rightarrow \ldots \]

Rules that produce this rewrite sequence

\[ \rho_1: \ f \rightarrow f \]
\[ \rho_2: \ f \rightarrow f \]
Consequences

Partial order convergence

\[ f \rightarrow f \rightarrow f \rightarrow f \rightarrow f \rightarrow \ldots \rightarrow \perp \]

Rules that produce this rewrite sequence

\[ \rho_1 : f \rightarrow f \]

\[ \rho_2 : f \rightarrow f \]
Consequences

Partial order convergence

\[ f \rightarrow f \rightarrow f \rightarrow f \rightarrow \ldots \rightarrow \perp \]

Rules that produce this rewrite sequence

\[ \rho_1 : f \rightarrow f \quad \rho_2 : f \rightarrow f \]

Theorem

Let \( S \) be a reduction in a GRS \( \mathcal{R} \):

\[ S : g \xrightarrow{m}{\mathcal{R}} h \quad \iff \quad S : g \xrightarrow{p}{\mathcal{R}} h \text{ total} \]
Examples

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$
Examples

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

\[
\begin{array}{c}
\bot \\
\Rightarrow \\
0 \Downarrow \\
\Rightarrow \\
0 :: \\
\Rightarrow \\
\ldots \\
\Downarrow \\
0 :: \\
\Rightarrow \\
0 :: s \\
\Rightarrow \\
\ldots \\
\Downarrow \\
0 :: s \\
\Rightarrow \\
\ldots \\
\Downarrow \\
0 :: s \\
\Rightarrow \\
\ldots
\end{array}
\]
Examples

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

Term graph rewriting with $h(x, y) \rightarrow h(y, x)$
Examples

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

```
\[
\begin{array}{cccc}
  & & &
  \downarrow & \rightarrow & \downarrow & \rightarrow & \cdots & \downarrow & \\
  & & & & & & & & &
  & & 0 & \downarrow & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
  & & & & & & & & &
  & & & & & s & \rightarrow & & s & \\
  & & & & & & & & &
  & & & & & & & \downarrow & & \downarrow & \\
  & & & & & & & & &
  & & & & & & & c & & c \\
\end{array}
\]
```

Term graph rewriting with $h(x, y) \rightarrow h(y, x)$

```
\[
\begin{array}{cccc}
  & & &
  f & \rightarrow & f \\
  & & & & & & & & &
  h & g & \rightarrow & h & g \\
  & & & & & & & & &
  c & \rightarrow & c & \rightarrow & c \\
\end{array}
\]
```
Examples

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

Term graph rewriting with $h(x, y) \rightarrow h(y, x)$
Examples

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

```
0 ⊥ 0 :: s ⊥ 0 :: s
```

Term graph rewriting with $h(x, y) \rightarrow h(y, x)$

```
f g h
  \downarrow  \downarrow
  f  h
  \downarrow  \downarrow
  g  c
```

```
f g h
  \downarrow  \downarrow
  f  h
  \downarrow  \downarrow
  g  c
```

```
f g h
  \downarrow  \downarrow
  f  h
  \downarrow  \downarrow
  g  c
```
Examples

Term graph rewriting with $\text{from}(x) \rightarrow x :: \text{from}(s(x))$

Term graph rewriting with $h(x, y) \rightarrow h(y, x)$
Metric vs. Partial Order Approach

Theorem (Soundness of metric convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\begin{array}{c}
\mathcal{R} \\
U(\cdot) \\
U(\mathcal{R})
\end{array}
\xrightarrow{g}
\xrightarrow{m}
\rightarrow
\begin{array}{c}
h \\
\end{array}
\]

Theorem (Completeness of partial order convergence)

For every orthogonal, left-finite GRS $\mathcal{R}$ we have

\[
\begin{array}{c}
\mathcal{R} \\
U(\cdot) \\
U(\mathcal{R})
\end{array}
\xrightarrow{g}
\xrightarrow{m}
\rightarrow
\begin{array}{c}
h \\
\end{array}
\]

\[U(\cdot)\]
Metric vs. Partial Order Approach

Theorem (Soundness of metric convergence)
For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\mathcal{R} \xrightarrow{g} m \xrightarrow{h} \mathcal{R}
\]

Theorem (Completeness of partial order convergence)
For every orthogonal, left-finite GRS $\mathcal{R}$ we have

\[
\mathcal{U}(\cdot) \downarrow \mathcal{U}(\cdot) \quad \mathcal{U}(\mathcal{R}) \xrightarrow{s} m \xrightarrow{t} \mathcal{U}(\mathcal{R})
\]
Metric vs. Partial Order Approach

Theorem (Soundness of partial order convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\begin{align*}
\mathcal{R} \quad &g \quad p \quad \rightarrow \quad h \\
\mathcal{U}(\cdot) \quad &s \quad p \quad \rightarrow \quad t \\
\mathcal{U}(\mathcal{R}) \quad &
\end{align*}
\]
Metric vs. Partial Order Approach

Theorem (Soundness of partial order convergence)
For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[ \overrightarrow{\mathcal{R}} \xrightarrow{g} p \rightarrow h \]
\[ \mathcal{U}(\cdot) \]
\[ \mathcal{U}(\mathcal{R}) \xrightarrow{s} p \rightarrow t \]

Theorem (Completeness of partial order convergence)
For every orthogonal, left-finite GRS $\mathcal{R}$ we have

\[ \mathcal{U}(\mathcal{R}) \xrightarrow{s} p \rightarrow t \]
\[ \mathcal{U}(\cdot) \]
\[ \mathcal{U}(\cdot) \]
\[ \overrightarrow{\mathcal{R}} \xrightarrow{g} \]
Metric vs. Partial Order Approach

Theorem (Soundness of partial order convergence)

For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\begin{align*}
\mathcal{R} & \quad g \\ 
\mathcal{U}(\cdot) & \quad \downarrow \\
\mathcal{U}(\mathcal{R}) & \quad s \\
\end{align*}
\]

\[
\begin{align*}
p & \quad \rightarrow \\
\mathcal{U}(\cdot) & \quad \downarrow \\
t & \quad \\
\end{align*}
\]

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\mathcal{U}(\cdot) & \quad \downarrow \\
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p & \quad \rightarrow \\
h & \quad \\
\end{align*}
\]
Metric vs. Partial Order Approach

**Theorem (Soundness of partial order convergence)**

For every left-linear, left-finite GRS $\mathcal{R}$ we have

\[
\begin{align*}
\mathcal{R} & \quad g \\
\mathcal{U}(\cdot) & \quad \mathcal{U}(\cdot) \\
\mathcal{U}(& \mathcal{R}) \quad s
\end{align*}
\]

\[
\begin{align*}
\rightarrow & \quad p \\
\rightarrow & \quad h
\end{align*}
\]

**Theorem (Completeness of partial order convergence)**

For every orthogonal, left-finite GRS $\mathcal{R}$ we have

\[
\begin{align*}
\mathcal{U}(\mathcal{R}) & \quad s \\
\mathcal{U}(\cdot) & \quad \mathcal{U}(\cdot) \\
\mathcal{U}(\cdot) & \quad \mathcal{U}(\cdot) \\
\mathcal{R} & \quad g
\end{align*}
\]

\[
\begin{align*}
\rightarrow & \quad p \\
\rightarrow & \quad t \\
\rightarrow & \quad t'
\end{align*}
\]
Conclusions

Simple structures formalising convergence on term graphs

- **intuitive & simple** generalisation of term rewriting counterparts
- the structures are "complete"
- "soundness" of limit & limit inferior (i.e. commutes with unravelling)
- But: weak partial order convergence is somewhat odd
## Conclusions

**Simple structures formalising convergence on term graphs**
- **intuitive & simple** generalisation of term rewriting counterparts
- the structures are “complete”
- “soundness” of limit & limit inferior (i.e. commutes with unravelling)
- But: weak partial order convergence is somewhat odd

**Strong convergence**
- regain **correspondence** between metric and partial order convergence
- soundness and completeness w.r.t. infinitary term rewriting